Dùng bất đẳng thức Côsi giải các bài toán sau:

Bài 1. Cho x, y, z > 0. Chứng minh: $P=\frac{x}{2x+y+z}+\frac{y}{x+2y+z}+\frac{z}{x+y+2z}\le\frac{3}{4}.$

Giải

Đặt: $\begin{cases}a=2x+y+z\\ b=x+2y+z\\ c=x+y+2z\end{cases}\Rightarrow a,b,c>0; x=\frac{3a-b-c}{4}; y=\frac{3b-a-c}{4}; z=\frac{3c-a-b}{4}$

BĐT cần chứng minh: $\frac{3a-b-c}{4a}+\frac{3b-c-a}{4b}+\frac{3c-a-b}{4c}\le\frac{3}{4}$

$$P=\frac{3a-b-c}{4a}+\frac{3b-c-a}{4b}+\frac{3c-a-b}{4c}=\frac{9}{4}-\frac{1}{4}\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)$$

$$\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}=\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\ge 6$$

$P\le\frac{9}{4}-\frac{6}{4}=\frac{3}{4}.$ Dấu "=" xảy ra x = y = z = 1.

Bài 2. Cho x, y, z > 0 và xyz = xy + yz + zx. Chứng minh:

$$P=\frac{1}{x+2y+3z}+\frac{1}{2x+3y+z}+\frac{1}{3x+y+2z}<\frac{3}{16}.$$

Giải

$$xy+yz+zx=xyz\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

$$1=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge 3\sqrt[3]{\frac{1}{xyz}}\Rightarrow\frac{1}{\sqrt[3]{xyz}}\le\frac{1}{3}$$

$$x+2y+3z\ge 3\sqrt[3]{6xyz}\Rightarrow\frac{1}{x+2y+3z}\le\frac{1}{3\sqrt[3]{6}}.\frac{1}{\sqrt[3]{xyz}}\le\frac{1}{9\sqrt[3]{6}}<\frac{1}{16}\qquad(1)$$

$$2x+3y+z\ge 3\sqrt[3]{6xyz}\Rightarrow\frac{1}{2x+3y+z}\le\frac{1}{3\sqrt[3]{6}}.\frac{1}{\sqrt[3]{xyz}}\le\frac{1}{9\sqrt[3]{6}}<\frac{1}{16}\qquad(2)$$

$$3x+y+3z\ge 3\sqrt[3]{6xyz}\Rightarrow\frac{1}{3x+y+3z}\le\frac{1}{3\sqrt[3]{6}}.\frac{1}{\sqrt[3]{xyz}}\le\frac{1}{9\sqrt[3]{6}}<\frac{1}{16}\qquad(3)$$

$$(1),(2),(3)\Rightarrow P<\frac{3}{16}.$$

Bài 3. Cho x, y, z > 0 và $x^2+y^2+z^2=1.$ Chứng minh:

$$\frac{x}{y^2+z^2}+\frac{y}{z^2+x^2}+\frac{z}{x^2+y^2}\ge\frac{3\sqrt{3}}{2}.$$

Giải

$$x^2+y^2+z^2=1\Rightarrow 0<x,y,z<1;\ y^2+z^2=1-x^2;\ z^2+x^2=1-y^2;\ x^2+y^2=1-z^2$$

$$P=\frac{x}{y^2+z^2}+\frac{y}{z^2+x^2}+\frac{z}{x^2+y^2}=\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$$

$$0<x<1\Rightarrow 1-x^2>0$$

$$\frac{2}{3}=\frac{2x^2+\left(1-x^2\right)+\left(1-x^2\right)}{3}\ge\sqrt[3]{2x^2\left(1-x^2\right)^2}$$

$$\Rightarrow\frac{8}{27}\ge 2x^2\left(1-x^2\right)^2\Rightarrow\frac{2}{3\sqrt{3}}\ge x\left(1-x^2\right)\Rightarrow\frac{2}{1-x^2}\ge 3\sqrt{3}x\Rightarrow\frac{x}{1-x^2}\ge\frac{3\sqrt{3}}{2}x^2\quad(1)$$

$$\frac{2}{3}=\frac{2y^2+\left(1-y^2\right)+\left(1-y^2\right)}{3}\ge\sqrt[3]{2y^2\left(1-y^2\right)^2}$$

$$\Rightarrow\frac{8}{27}\ge 2y^2\left(1-y^2\right)^2\Rightarrow\frac{2}{3\sqrt{3}}\ge y\left(1-y^2\right)\Rightarrow\frac{2}{1-y^2}\ge 3\sqrt{3}y\Rightarrow\frac{y}{1-y^2}\ge\frac{3\sqrt{3}}{2}y^2\quad(2)$$

$$\frac{2}{3}=\frac{2z^2+\left(1-z^2\right)+\left(1-z^2\right)}{3}\ge\sqrt[3]{2z^2\left(1-z^2\right)^2}$$

$$\Rightarrow\frac{8}{27}\ge 2z^2\left(1-z^2\right)^2\Rightarrow\frac{2}{3\sqrt{3}}\ge z\left(1-z^2\right)\Rightarrow\frac{2}{1-z^2}\ge 3\sqrt{3}z\Rightarrow\frac{z}{1-z^2}\ge\frac{3\sqrt{3}}{2}z^2\quad(3)$$

$(1),(2),(3)\Rightarrow P\ge\frac{3\sqrt{3}}{2}\left(x^2+y^2+z^2\right)=\frac{3\sqrt{3}}{2}$, dấu "=" xảy ra $\Leftrightarrow x=y=z=\frac{1}{\sqrt{3}}$.

Bài 4. Cho x, y > 0 và x + y = 1. Chứng minh: $\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}\ge\sqrt{2}$.

Giải

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}=\frac{x}{\sqrt{1-x}}+\frac{1-x}{\sqrt{1-(1-x)}}=\frac{x}{\sqrt{1-x}}+\frac{1-x}{\sqrt{x}}\ge 2\sqrt{\frac{x(1-x)}{\sqrt{x}\sqrt{1-x}}}=2\sqrt[4]{x(1-x)}$$

$$x(1-x)=-x^2+x=-\left(x^2-x\right)=-\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right]=\frac{1}{4}-\left(x-\frac{1}{2}\right)^2\ge\frac{1}{4}$$

$$2\sqrt[4]{x(1-x)}\ge 2.\frac{1}{\sqrt[4]{4}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}\ge\sqrt{2}$. Dấu "=" xảy ra $\Leftrightarrow x=y=\frac{1}{2}$.

Bài 5. Cho a, b, c > 0 và $a^4+b^4+c^4=48$. Chứng minh: $ab^2+bc^2+ca^2\le 24$.

Giải

$$P=ab^2+bc^2+ca^2\Rightarrow 2P=2ab^2+2bc^2+2ca^2$$

$$2ab^2=2bab\le\left(\frac{b+2}{2}\right)^2\left(\frac{a+b}{2}\right)^2=\frac{(b+2)^2}{4}.\frac{(a+b)^2}{4}=\frac{\left(ab+b^2+2a+2b\right)^2}{16}$$

$$ab\le\frac{a^2+b^2}{2};\ 2a\le\frac{a^2+4}{2};\ 2b\le\frac{b^2+4}{2}\Rightarrow ab+b^2+2a+2b\le a^2+2b^2+4$$

$$\left(ab+b^2+2a+2b\right)^2\le\left(a^2+2b^2+4\right)^2=a^4+4b^4+16+4a^2b^2+8a^2+16b^2$$

$$4a^2b^2\le 2a^4+2b^4;\ 8a^2\le a^4+16;\ 16b^2\le 2b^4+32$$

$$a^4+4b^4+16+4a^2b^2+8a^2+16b^2\le a^4+4b^4+16+2a^4+2b^4+a^4+16+2b^4+32$$

$$a^4+4b^4+256+4a^2b^2+8a^2+16b^2\le 4a^4+8b^4+64$$

$$2ab^2\le\frac{a^4+2b^4+16}{4}$$

$$2bc^2\le\frac{b^4+2c^4+16}{4};\ 2ca^2\le\frac{c^4+2a^4+16}{4}$$

$2P\le\frac{3\left(a^4+b^4+c^4\right)+48}{4}=\frac{4.48}{4}\Rightarrow P\le 24$. Dấu "=" xảy ra $\Leftrightarrow$ a = b = c = 2.

Bài 6. Cho x, y, z > 0 và x + y + z = 1. Chứng minh: $\sqrt{1-x}+\sqrt{1-y}+\sqrt{1-z}\le\sqrt{6}$.

Giải

$$x+y+z=1\Rightarrow 1-x=y+z;\ 1-y=z+x;\ 1-z=x+y$$

$$P=\sqrt{1-x}+\sqrt{1-y}+\sqrt{1-z}\le\sqrt{6}\Leftrightarrow\sqrt{y+z}+\sqrt{z+x}+\sqrt{x+y}\le\sqrt{6}$$

$$\sqrt{y+z}=\sqrt{\frac{3}{2}}\sqrt{(y+z).\frac{2}{3}}\le\sqrt{\frac{3}{2}}.\frac{y+z+\frac{2}{3}}{2}=\sqrt{\frac{3}{2}}.\frac{3y+3z+2}{6}\qquad(1)$$

$$\sqrt{z+x}=\sqrt{\frac{3}{2}}\sqrt{(z+x).\frac{2}{3}}\le\sqrt{\frac{3}{2}}.\frac{z+x+\frac{2}{3}}{2}=\sqrt{\frac{3}{2}}.\frac{3z+3x+2}{6}\qquad(2)$$

$$\sqrt{x+y}=\sqrt{\frac{3}{2}}\sqrt{(x+y).\frac{2}{3}}\le\sqrt{\frac{3}{2}}.\frac{x+y+\frac{2}{3}}{2}=\sqrt{\frac{3}{2}}.\frac{3x+3y+2}{6}\qquad(3)$$

$(1)+(2)+(3): P\le\frac{\sqrt{3}}{\sqrt{2}}(x+y+z+1)=\frac{2\sqrt{3}}{\sqrt{2}}=\sqrt{6}.$ Dấu "=" xảy ra $\Leftrightarrow x=y=z=\frac{1}{3}.$

Bài 7. Cho x, y, z > 0 và xyz = 1. Chứng minh: $\frac{x^2}{1+y}+\frac{y^2}{1+z}+\frac{z^2}{1+x}\ge\frac{3}{2}.$

Giải

$$P=\frac{x^2}{1+y}+\frac{y^2}{1+z}+\frac{z^2}{1+x}$$

$$1+y\ge2\sqrt{y}\Rightarrow\frac{x^2}{1+y}\ge\frac{x^2}{2\sqrt{y}}$$

$$1+z\ge2\sqrt{z}\Rightarrow\frac{y^2}{1+z}\ge\frac{y^2}{2\sqrt{z}}$$

$$1+x\ge2\sqrt{x}\Rightarrow\frac{z^2}{1+x}\ge\frac{z^2}{2\sqrt{x}}$$

$P\ge\frac{1}{2}\left(\frac{x^2}{\sqrt{y}}+\frac{y^2}{\sqrt{z}}+\frac{z^2}{\sqrt{x}}\right)\ge\frac{3}{2}\sqrt[3]{\frac{x^2y^2z^2}{\sqrt{xyz}}}=\frac{3}{2}.$ Dấu "=" xảy ra $\Leftrightarrow x=y=z=1$

Bài 8. Cho x, y, z là 3 số không âm và x + y + z = 0. Chứng minh $\sqrt{2+4^x}+\sqrt{2+4^y}+\sqrt{2+4^z}\ge3\sqrt{3}.$

Giải

$$P=\sqrt{2+4^x}+\sqrt{2+4^y}+\sqrt{2+4^z}$$

$$\sqrt{2+4^x}=\frac{1}{\sqrt{3}}\sqrt{(2+4^x).3}\ge\frac{1}{\sqrt{3}}.\frac{2+4^x+3}{2}$$

$$\sqrt{2+4^y}=\frac{1}{\sqrt{3}}\sqrt{(2+4^y).3}\ge\frac{1}{\sqrt{3}}.\frac{2+4^y+3}{2}$$

$$\sqrt{2+4^z}=\frac{1}{\sqrt{3}}\sqrt{(2+4^z).3}\ge\frac{1}{\sqrt{3}}.\frac{2+4^z+3}{2}$$

$$P\ge\frac{1}{2\sqrt{3}}(15+4^x+4^y+4^z)\ge\frac{1}{2\sqrt{3}}\left(15+3\sqrt[3]{4^{x+y+z}}\right)=\frac{1}{2\sqrt{3}}(15+3)=3\sqrt{3}.$$

Dấu "=" xảy ra $\Leftrightarrow x=y=z=0.$

Bài 9. Cho x, y, z > 0 và $x+y+z\ge3.$ Chứng minh $\frac{x}{\sqrt{y}}+\frac{y}{\sqrt{z}}+\frac{z}{\sqrt{x}}\ge3.$

Giải

Đặt: $a=\sqrt{x},b=\sqrt{y},c=\sqrt{z}\Rightarrow a,b,c>0: x+y+z\ge3\Leftrightarrow a^2+b^2+c^2\ge3.$

BĐT cần chứng minh: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge3$

$$P=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^3c}{abc}+\frac{b^3a}{abc}+\frac{c^3b}{abc}=\frac{a^3c+c^3b+b^3a}{abc}\ge\frac{3\sqrt[3]{a^3b^3c^3}}{abc}=3.$$

Dấu "=" xảy ra $\Leftrightarrow a=b=c=1\Leftrightarrow x=y=z=1.$